WEIGHTED ESTIMATES FOR THE MULTISUBLINEAR MAXIMAL FUNCTION

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ABSTRACT. A formulation of the Carleson embedding theorem in the multilinear setting is proved which allows to obtain a multilinear analogue of Sawyer's two weight theorem for the multisublinear maximal function $\mathcal{M}$ introduced in [8]. A multilinear version of the B_p theorem from [6] is also obtained and a mixed $A_{\vec{P}} - W^{\infty}_{\vec{P}}$ bound for $\mathcal{M}$ is proved as well.

1. Introduction

The beginning of the modern theory of weights was originated in the works of R. Hunt, B. Muckenhoupt, R. Wheeden, R. Coifman and C. Fefferman in the decade of the 70's. In [12] B. Muckenhoupt characterized the class of weights u, v for which the following weak inequality holds

$$\sup_{\lambda>0} \lambda^p \int_{\{Mf>\lambda\}} u(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x)dx,\ f \in L^p(v), \tag{1.1}$$

where M denotes the Hardy–Littlewood maximal operator and $p \geq 1$. This condition on the weights is known as A_p condition, namely

$$[u,v]_{A_p} := \sup_Q \left(\frac{1}{|Q|}\int_Q u(x)dx\right)\left(\frac{1}{|Q|}\int_Q v(x)^{-\frac{1}{p-1}}\right)^{p-1} < \infty,\ p > 1$$

where the supremum is taken over all the cubes in $\mathbb{R}^n$. When $p = 1$, $(\int_Q \frac{1}{|Q|} v(x)^{-\frac{1}{p-1}})^{p-1}$ must be understood as $(\inf_Q v)^{-1}$. In the particular case when $u = v$, Muckenhoupt also proved that the following strong estimate

$$\int_{\mathbb{R}^n} (Mf(x))^p v(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x)dx,\ f \in L^p(v),$$

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holds if and only if v satisfies the A_p condition. However, the problem of finding a condition on the weights u, v satisfying the strong estimate above was more complicated. In [13] E. Sawyer characterized the two weight inequality, showing that $M : L^p(v) \longrightarrow L^p(u)$ if and only if the pair (u, v) satisfies the following testing condition known as Sawyer's S_p condition

$$[u, v]_{S_p} = \sup_Q \left(\frac{\int_Q M(\chi_Q \sigma)^p u dx}{\sigma(Q)} \right)^{1/p} < \infty, \tag{1.2}$$

where $\sigma = v^{1-p'}$ and $1 < p < \infty$. Motivated by these results the theory of weighted inequalities developed rapidly, not only for the Hardy–Littlewood maximal operator but also for some of the main operators in Harmonic Analysis like Calderón–Zygmund operators. Much later the interest of the focused in determining the sharp dependence of the $L^p(w)$ operator norm in term of the relevant constant involving the weights.

On this point, the problem for the Hardy–Littlewood maximal operator was solved by S. Buckley [1] who proved

$$||M(f)||_{L^p(w)} \leq C\, p'\, [w]_{A_p}^{\frac{1}{p-1}}, \tag{1.3}$$

where C is a dimensional constant. Motivated by this result and others, K. Moen found in [11] a quantitative form of E. Sawyer's result mentioned above in terms of Sawyer's condition (1.2), namely

$$||M||_{L^p(v) \longrightarrow L^p(u)} \approx [u, v]_{S_p}. \tag{1.4}$$

Recently, Hytönen and C. Pérez in [6] (see also [7] for a better result and a simplified proof) improved Buckley's bound (1.3) replacing a portion of the A_p constant by the weaker A_∞ constant as defined by Fujii in [4] and later used in the work of Wilson [14]. The A_∞ constant is defined as follows

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q), \tag{1.5}$$

where the supremum is taken over all the cubes Q in $\mathbb{R}^n$. In [6] the authors show in a two-weight setting and for $p > 1$ that

$$||M(f\sigma)||_{L^p(w)} \leq Cp'(B_p[w, \sigma])^{1/p} ||f||_{L^p(\sigma)}, \tag{1.6}$$

and

$$||M(f\sigma)||_{L^p(w)} \leq Cp'([w]_{A_p}[\sigma]_{A_\infty})^{1/p} ||f||_{L^p(\sigma)}, \tag{1.7}$$

where C in both inequalities is a dimensional constant and

$$B_p[w,\sigma] := \sup_Q \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q \sigma\right)^p \exp\left(\frac{1}{|Q|}\int_Q \log\sigma^{-1}\right), \tag{1.8}$$

is known as the B_p constant of the weights w and σ. This constant clearly satisfies

$$[w]_{A_p} \leq B_p[w,\sigma] \leq [w]_{A_p}[\sigma]'_{A_\infty},$$

where $[\sigma]'_{A_\infty}$ denotes the A_∞ constant introduced by Hruščěv in[5] defined as follows

$$[w]'_{A_\infty} = \sup_Q \left(\frac{1}{|Q|}\int_Q w\right)\exp\left(\frac{1}{|Q|}\int_Q \log w^{-1}\right).$$

In the one weight setting and by a standard change-of-weight argument, (1.7) implies

$$||M||_{L^p(w)} \leq Cp'([w]_{A_p}[\sigma]_{A_\infty})^{1/p}, \tag{1.9}$$

where $\sigma = w^{1-p'}$.

The aim of this article is to give some multilinear analogues of some of the above mentioned results following the spirit of the theory of multiple weights developed in [8]. One of the main objects of this new theory is the following extension of the classical Hardy–Littlewood maximal function. Given $\vec{f} = (f_1,\dots,f_m)$, we define following [8] the multi(sub)linear maximal operator $\mathcal{M}$ by

$$\mathcal{M}(\vec{f})(x) = \sup_{Q\ni x}\prod_{i=1}^m \frac{1}{|Q|}\int_Q |f_i(y_i)|dy_i,$$

where the supremum is taken over all cubes Q containing x. The importance of this operator stems from the fact that it controls the class of multilinear Calderón–Zygmund operators as it is shown in [8]. A particular example of this relationship is the class of weights characterizing the weighted L^p spaces for which both operators are bounded. To define this class of weights we let $\vec{w} = (w_1,\dots,w_m)$ and $\vec{P} = (p_1,\dots,p_m)$. Set $\frac{1}{p} = \frac{1}{p_1}+\cdots+\frac{1}{p_m}$ and $\nu_{\vec{w}} = \prod_{i=1}^m w_i^{p/p_i}$. We say that $\vec{w}$ satisfies the $A_{\vec{P}}$ condition if

$$[\vec{w}]_{A_{\vec{P}}} = \sup_Q \Big(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}\Big)\prod_{i=1}^m \Big(\frac{1}{|Q|}\int_Q w_i^{1-p_i'}\Big)^{p/p_i'} < \infty.$$

It is easy to see that in the linear case (that is, if $m=1$) $[\vec{w}]_{A_{\vec{P}}} = [w]_{A_p}$ is the usual A_p constant. In [8] the following multilinear extension of the Muckenhoupt A_p theorem for the maximal function was obtained: the inequality

$$\|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C\prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)} \tag{1.10}$$

holds for every $\vec{f}$ if and only if $\vec{w}$ satisfies the $A_{\vec{P}}$ condition.

Very recently in [3] A. Lerner, C. Pérez and the second author proved a multilinear version of Buckley's result as well as a full analogue of (1.7). In this work the authors found that the multilinear version of (1.7) is sharp when $m \geq 1$, although is much more complicated to do the same for Buckley's result. In this case, several partial results were obtained in [3] which have been improved in [9].

Next we state the main tool of this paper. This lemma extends to the multilinear setting a nonstandard formulation of the (dyadic) Carleson embedding theorem proved in [6] and it will allow us to prove our main results.

Lemma 1.1. *Suppose that the nonnegative numbers $\{a_Q\}_Q$ satisfy*

$$\sum_{Q\subset R} a_Q \leq A \int_R \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}}\, dx,\ \forall R \in \mathscr{D} \tag{1.11}$$

where σ_i are weights for $i = 1, \ldots, m$. Then for all $1 < p_i < \infty$ and $p \in (1, \infty)$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and for all $f_i \in L^{p_i}(\sigma_i)$,

$$\begin{aligned}\left(\sum_{Q\in\mathscr{D}} a_Q \Big(\prod_{i=1}^m \frac{1}{\sigma_i(Q)} \int_Q f_i(y_i)\sigma_i(y_i)dy_i\Big)^p\right)^{1/p} &\leq A||\mathcal{M}^d_{\vec{\sigma}}(\vec{f})||_{L^p(\nu_{\vec{\sigma}})} \\ &\leq A \prod_{i=1}^m p_i' ||f_i||_{L^{p_i}(\sigma_i)},\end{aligned} \tag{1.12}$$

where $\mathcal{M}^d_{\vec{\sigma}}(\vec{f}) = \sup\limits_{\substack{Q\ni x\\ Q\in\mathscr{D}}} \prod_{i=1}^m \frac{1}{\sigma_i(Q)} \int_Q |f_i(y_i)|\sigma_i(y_i)dy_i.$

Next we establish a generalization of Sawyer's theorem to the multilinear setting. Very recently it was shown in [10] a multilinear version of Sawyer's theorem using a kind of monotone property on the weights. We establish here another condition that is a sort of reverse Hölder inequality in the multilinear setting (see Section 2 for more details) and that was used by the first author in [2] in the setting of martingale spaces. When $m = 1$ this reverse Hölder condition is superfluous and we recover the linear result of Moen (1.4).

Theorem 1.2. *Let $1 < p_i < \infty$, $i = 1, \ldots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}$. Let v and w_i be weights. If we suppose that $\vec{w} \in RH_{\vec{P}}$ then there exists a positive constant C such that*

$$||\mathcal{M}(\overrightarrow{f\sigma})||_{L^p(v)} \leq C \prod_{i=1}^m ||f_i||_{L^{p_i}(\sigma_i)},\ \ f_i \in L^{p_i}(\sigma_i), \tag{1.13}$$

where $\sigma_i = w_i^{1-p_i'}$, if and only if $(v, \vec{w}) \in S_{\vec{P}}$. Moreover, if we denote the smallest constant C in (1.13) by $||\mathcal{M}||$, we obtain

$$[v,\vec{w}]_{S_{\vec{P}}} \lesssim ||\mathcal{M}|| \lesssim [v,\vec{w}]_{S_{\vec{P}}}[\vec{w}]_{RH_{\vec{P}}}^{1/p}. \tag{1.14}$$

Here we make some remarks related to the previous theorem.

Remark 1.3. In the particular case when $v = \nu_{\vec{w}}$, the following statements are equivalent:

(1) $\vec{w} \in A_{\vec{P}}$.
(2) $\sigma_i = w_i^{1-p_i'} \in A_{mp_i'}$, for $i = 1, \ldots, m$ and $\nu_{\vec{w}} \in A_{mp}$.
(3) $(\nu_{\vec{w}}, \vec{w}) \in S_{\vec{P}}$.
(4) There exists a positive constant C such that

$$||\mathcal{M}(\vec{f})||_{L^p(\nu_{\vec{w}})} \leq C \prod_{i=1}^{m} ||f_i||_{L^{p_i}(w_i)}, \;\; f_i \in L^{p_i}(w_i). \tag{1.15}$$

Indeed, the equivalence between (1), (2) and (4) was proved in [8, Th. 3.6, Th. 3.7]. It can be easily seen that in this particular case $[\nu_{\vec{w}}, \vec{w}]_{S_{\vec{P}}} \lesssim ||\mathcal{M}||$ where $||\mathcal{M}||$ denotes the smallest constant in (1.15) and $[\vec{w}]_{A_{\vec{P}}} \lesssim [\nu_{\vec{w}}, \vec{w}]_{S_{\vec{P}}}^{p}$. Therefore we have that (4) implies (3) and (3) implies (1). So we have obtained that all the statements are equivalent.

Additionally, following [3, Th. 1.1], we also have that $||\mathcal{M}|| \lesssim [\vec{w}]_{A_{\vec{P}}}^{1/p} \prod_{i=1}^{m} [\sigma_i]_{\infty}^{\frac{1}{p_i}}$. So, we have obtained

$$[\vec{w}]_{A_{\vec{P}}}^{1/p} \lesssim [v_{\vec{w}}, \vec{w}]_{S_{\vec{P}}} \lesssim ||\mathcal{M}|| \lesssim [\vec{w}]_{A_{\vec{P}}}^{1/p} \prod_{i=1}^{m} [\sigma_i]_{\infty}^{\frac{1}{p_i}}. \tag{1.16}$$

Remark 1.4. As we have observed in the previous remark, $RH_{\vec{P}}$ condition is not necessary when $v = \nu_{\vec{w}}$ in Theorem 1.2. We are not sure if this condition can be removed in the general case.

Now if we define an analogue of the B_p constant in the multilinear setting (see Section 2 for the definition), we obtain an extension of (1.6).

Theorem 1.5. *Let $1 < p_i < \infty$, $i = 1, \ldots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}$. Let v and w_i be weights. Then*

$$||\mathcal{M}(\overrightarrow{f\sigma})||_{L^p(v)} \lesssim [v, \vec{\sigma}]_{B_{\vec{P}}}^{1/p} \prod_{i=1}^{m} ||f_i||_{L^{p_i}(\sigma_i)}, \;\; f_i \in L^{p_i}(\sigma_i), \tag{1.17}$$

where $\sigma_i = w_i^{1-p_i'}$, $\vec{\sigma} = (\sigma_1, \ldots, \sigma_m)$ and $\overrightarrow{f\sigma} = (f_1\sigma_1, \ldots, f_m\sigma_m)$.

And finally, using a generalization of the Wilson–Fujii A_∞ constant denoted as $[\vec{w}]_{W_{\vec{P}}^{\infty}}$ and the two-weight constant $[v, \vec{w}]_{A_{\vec{P}}}$ (see Section 2 for the definitions), we get a mixed $A_{\vec{P}} - W_{\vec{P}}^{\infty}$ bound for $\mathcal{M}$ that extends (1.7) to the multilinear setting.

Theorem 1.6. *Let $1 < p_i < \infty$, $i = 1, \ldots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}$. Let v and w_i be weights. Then*

$$||\mathcal{M}(\overrightarrow{f\sigma})||_{L^p(v)} \lesssim ([v, \vec{w}]_{A_{\vec{P}}}[\vec{\sigma}]_{W^\infty_{\vec{P}}})^{1/p} \prod_{i=1}^{m} ||f_i||_{L^{p_i}(\sigma_i)}, \ f_i \in L^{p_i}(\sigma_i), \tag{1.18}$$

where $\sigma_i = w_i^{1-p_i'}$, $\vec{\sigma} = (\sigma_1, \ldots, \sigma_m)$ and $\overrightarrow{f\sigma} = (f_1\sigma_1, \ldots, f_m\sigma_m)$.

The paper is organized as follows. Some preliminary definitions are established in Section 2. In Section 3 we give all the proofs of our results.

Throughout this paper, we will use the notation $A \lesssim B$ to indicate that there is a constant c, independent of the weight constant, such that $A \leq cB$.

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2. Preliminaries

Before proving our main results, we first recall some definitions and results related to dyadic grids and some constants involved in the multiple theory of weights.

2.1. **Dyadic grids.** Recall that the standard dyadic grid $\mathcal{D}$ in $\mathbb{R}^n$ consists of the cubes

$$2^{-k}([0,1)^n + j), \quad k \in \mathbb{Z}, j \in \mathbb{Z}^n.$$

By a *general dyadic grid* $\mathscr{D}$ we mean a collection of cubes with the following properties:

1. for any $Q \in \mathscr{D}$ its sidelength ℓ_Q is of the form $2^k, k \in \mathbb{Z}$
2. $Q \cap R \in \{Q, R, \emptyset\}$ for any $Q, R \in \mathscr{D}$.
3. the cubes of a fixed sidelength 2^k form a partition of $\mathbb{R}^n$.

We say that $\{Q_j^k\}$ is a *sparse family* of cubes if:

1. the cubes Q_j^k are disjoint in j, with k fixed.
2. if $\Omega_k = \cup_j Q_j^k$, then $\Omega_{k+1} \subset \ \Omega_k$.
3. $|\Omega_{k+1} \cap Q_j^k| \leq \frac{1}{2}|Q_j^k|$.

With each sparse family $\{Q_j^k\}$ we associate the sets $E_j^k = Q_j^k \setminus \Omega_{k+1}$. Observe that the sets E_j^k are pairwise disjoint and $|Q_j^k| \leq 2|E_j^k|$.

In the sequel we will use the following lemmas that could be found in [6] and [3], respectively.

Lemma 2.1. *There are 2^n dyadic grids $\mathscr{D}_\alpha$ such that for any cube $Q \subset \mathbb{R}^n$ there exists a cube $Q_\alpha \in \mathscr{D}_\alpha$ such that $Q \subset Q_\alpha$ and $\ell_{Q_\alpha} \leq 6\ell_Q$.*

Lemma 2.2. *For any non-negative integrable $f_i, i = 1, \dots, m$, there exist sparse families $\mathcal{S}_\alpha \in \mathscr{D}_\alpha$ such that for all $x \in \mathbb{R}^n$,*

$$\mathcal{M}(\vec{f})(x) \leq (2 \cdot 12^n)^m \sum_{\alpha=1}^{2^n} \mathcal{A}_{\mathscr{D}_\alpha, \mathcal{S}_\alpha}(\vec{f})(x).$$

2.2. **Some constants on multiple weight theory.** Next we state the notation that we will follow in the sequel related to some constants involved in the multiple theory of weights. To define these constants, let $w_1, \dots, w_m$ and v be weights and let us denote $\vec{w} = (w_1, \dots, w_m)$. Also let $1 < p_1, \dots, p_m < \infty$ and p be numbers such that $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$ and denote $\vec{P} = (p_1, \dots, p_m)$.

We say that $(v, \vec{w})$ satisfies the $A_{\vec{P}}$ condition if

$$[v, \vec{w}]_{A_{\vec{P}}} := \sup_Q \Big(\frac{1}{|Q|}\int_Q v\Big) \prod_{i=1}^m \Big(\frac{1}{|Q|}\int_Q w_i^{1-p_i'}\Big)^{p/p_i'} < \infty. \tag{2.1}$$

When $v = \nu_{\vec{w}} := \prod_{i=1}^m w_i^{\frac{p}{p_i}}$, we will write $[\nu_{\vec{w}}, \vec{w}]_{A_{\vec{P}}}$ as $[\vec{w}]_{A_{\vec{P}}}$.

Next we define the multilinear analogues of the A_∞ constant defined by Fujii in [4], the B_p constant defined by Hytönen and Pérez in [6] and the S_p constant defined by Sawyer in [13], respectively. We say that

(1) $\vec{w}$ satisfies the $W_{\vec{P}}^\infty$ condition if

$$[\vec{w}]_{W_{\vec{P}}^\infty} = \sup_Q \Big(\int_Q \prod_{i=1}^m M(w_i\chi_Q)^{\frac{p}{p_i}} dx\Big)\Big(\int_Q \prod_{i=1}^m w_i^{\frac{p}{p_i}} dx\Big)^{-1} < \infty.$$

(2) $(v, \vec{w})$ satisfies the $B_{\vec{P}}$ condition if

$$[v, \vec{w}]_{B_{\vec{P}}} := \sup_Q \frac{v(Q)}{|Q|}\Big(\prod_{i=1}^m \frac{w_i(Q)}{|Q|}\Big)^p \exp\Big(\frac{1}{|Q|}\int_Q \log \prod_{i=1}^m w_i^{-\frac{p}{p_i}} dx\Big) < \infty.$$

(3) $(v, \vec{w})$ satisfies the $S_{\vec{P}}$ condition if

$$[v, \vec{w}]_{S_{\vec{P}}} = \sup_Q \Big(\int_Q \mathcal{M}(\overrightarrow{\sigma\chi_Q})^p v dx\Big)^{\frac{1}{p}} \Big(\prod_{i=1}^m \sigma_i(Q)^{\frac{1}{p_i}}\Big)^{-1} < \infty,$$

where $\overrightarrow{\sigma\chi_Q} = (\sigma_1\chi_Q, \dots, \sigma_m\chi_Q)$ and $\sigma_i = w_i^{1-p_i'}$ for all $i = 1, \dots, m$ and all the suprema in the above definitions are taken over all cubes Q in $\mathbb{R}^n$.

Additionally we define a multiple Reverse Hölder condition that we will use in the following. We say that $\vec{w}$ satisfies the $RH_{\vec{P}}$ condition if there exists a positive constant C such that

$$\prod_{i=1}^m \Big(\int_Q \sigma_i dx\Big)^{\frac{p}{p_i}} \leq C \int_Q \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx, \tag{2.2}$$

where $\sigma_i = w_i^{1-p_i'}$ for $i = 1, \dots, m$. We denote by $[\vec{w}]_{RH_{\vec{P}}}$ the smallest constant C in (2.2).

3. Proofs

We start proving Lemma 1.1, that is, the multilinear version of the dyadic Carleson embedding theorem. It follows a scheme of proof similar to the one used by Hytönen and Pérez in [6].

Proof of Lemma 1.1. Let us see the sum

$$\sum_{Q\in\mathscr{D}} a_Q \left(\prod_{i=1}^m \frac{1}{\sigma_i(Q)} \int_Q f_i(y_i)\sigma_i(y_i)dy_i\right)^p$$

as an integral on a measure space $(\mathscr{D}, 2^{\mathscr{D}}, \mu)$ built over the set of dyadic cubes $\mathscr{D}$, assigning to each $Q \in \mathscr{D}$ the measure a_Q. Thus

$$\begin{aligned}
&\sum_{Q\in\mathscr{D}} a_Q \left(\prod_{i=1}^m \frac{1}{\sigma_i(Q)} \int_Q f_i(y_i)\sigma_i(y_i)dy_i\right)^p = \\
&= \int_0^\infty p\lambda^{p-1}\mu\left\{Q \in \mathscr{D} : \prod_{i=1}^m \frac{1}{\sigma_i(Q)} \int_Q f_i(y_i)\sigma_i(y_i)dy_i > \lambda\right\} \\
&=: \int_0^\infty p\lambda^{p-1}\mu(\mathscr{D}_\lambda)d\lambda.
\end{aligned}$$

Let us denote by $\mathscr{D}_\lambda^*$ the set of maximal dyadic cubes R with the property that $\prod_{i=1}^m \frac{1}{\sigma_i(Q)} \int_R f_i(y_i)\sigma_i(y_i)dy_i > \lambda$. Then the cubes $R \in \mathscr{D}_\lambda^*$ are disjoint and their union is equal to the set $\{\mathcal{M}_{\vec{\sigma}}^d(\vec{f}) > \lambda\}$. Thus

$$\begin{aligned}
\mu(\mathscr{D}_\lambda) = \sum_{Q\in\mathscr{D}_\lambda} a_Q &\le \sum_{R\in\mathscr{D}_\lambda^*} \sum_{Q\subset R} a_Q \\
&\le A \sum_{R\in\mathscr{D}_\lambda^*} \int_R \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx \\
&= A \int_{\{\mathcal{M}_{\vec{\sigma}}^d(\vec{f})>\lambda\}} \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx.
\end{aligned}$$

Then we obtain

$$\begin{aligned}\sum_{Q\in\mathscr{D}} a_Q \left(\prod_{i=1}^m \frac{1}{\sigma_i(Q)}\int_Q f_i(y_i)\sigma_i(y_i)dy_i\right)^p &\le A\int_0^\infty p\lambda^{p-1}\int_{\{\mathcal{M}^d_{\vec\sigma}(\vec f)>\lambda\}}\prod_{i=1}^m \sigma_i^{\frac{p}{p_i}}dxd\lambda\\ &= A\int_{\mathbb{R}^n}\mathcal{M}^d_{\vec\sigma}(\vec f)^p\prod_{i=1}^m\sigma_i^{\frac{p}{p_i}}dx\\ &\le A\int_{\mathbb{R}^n}\prod_{i=1}^m((M^d_{\sigma_i}(f_i))^{p_i}\sigma_i)^{\frac{p}{p_i}}dx\\ &\le A\prod_{i=1}^m\left(\int_{\mathbb{R}^n}(M^d_{\sigma_i}(f_i))^{p_i}\sigma_i dx\right)^{\frac{p}{p_i}}\\ &\le A\prod_{i=1}^m (p_i')^p\left(\int_{\mathbb{R}^n}|f_i|^{p_i}\sigma_i dx\right)^{\frac{p}{p_i}},\end{aligned}$$

where we have used that $\mathcal{M}^d_{\vec\sigma}(\vec f)\le\prod_{i=1}^m M^d_{\sigma_i}(f_i)$, Hölder's inequality and the boundedness properties of $M^d_{\sigma_i}(f_i)$ in $L^{p_i}(\sigma_i)$. □

Next we prove Theorem 1.2 making use of Lemma 1.1.

Proof of Theorem 1.2. It is clear that (1.13) implies the $S_{\vec P}$ condition without using that $(v,\vec w)\in RH_{\vec P}$. Thus, it remains to prove that $(v,\vec w)\in S_{\vec P}$ implies (1.13) to complete the proof of the theorem.

By Lemma 2.1, it suffices to prove the theorem for the dyadic maximal operators $\mathcal{M}^{\mathscr{D}_\alpha}$. Since the proof is independent of the particular dyadic grid, without loss of generality we consider $\mathcal{M}^d$ taken with respect to the standard dyadic grid $\mathcal{D}$. Next we proceed as in the proof of Lemma 2.2. Let $a=2^{m(n+1)}$ and for $k\in\mathbb{Z}$ consider the following sets

$$\Omega_k=\{x\in\mathbb{R}^n:\mathcal{M}^d(\overrightarrow{f\sigma})>a^k\}.$$

Then we have that $\Omega_k=\cup_j Q_j^k$, where the cubes Q_j^k are pairwise disjoint with k fixed, and

$$a^k<\prod_{i=1}^m\frac{1}{|Q_j^k|}\int_{Q_j^k}|f_i(y_i)|\sigma_i(y_i)dy_i\le 2^{mn}a^k.$$

It follows that

$$\begin{aligned}\int_{\mathbb{R}^n} \mathcal{M}^d(\overrightarrow{f\sigma})^p v dx &= \sum_k \int_{\Omega_k \setminus \Omega_{k+1}} \mathcal{M}^d(\overrightarrow{f\sigma})^p v dx \\ &\leq a^p \sum_k \int_{\Omega_k \setminus \Omega_{k+1}} a^{kp} v dx \\ &= a^p \sum_{k,j} a^{kp} v(E_j^k),\end{aligned}$$

since $\Omega_k \setminus \Omega_{k+1} = \cup_j E_j^k$ where the sets E_j^k are the sets associated with the family $\{Q_j^k\}$. Then, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \mathcal{M}^d(\overrightarrow{f\sigma})^p v dx &\leq a^p \sum_{k,j} \left(\prod_{i=1}^m \frac{1}{|Q_j^k|} \int_{Q_j^k} |f_i| \sigma_i dy_i \right)^p v(E_j^k) \\ &= a^p \sum_{k,j} v(E_j^k) \left(\prod_{i=1}^m \frac{\sigma_i(Q_j^k)}{|Q_j^k|} \right)^p \left(\prod_{i=1}^m \frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} |f_i| \sigma_i dy_i \right)^p \\ &= a^p \sum_{Q \in \mathcal{D}} a_Q \left(\prod_{i=1}^m \frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} |f_i| \sigma_i dy_i \right)^p,\end{aligned}$$

where $a_Q = v(E(Q)) \left(\prod_{i=1}^m \frac{\sigma_i(Q)}{|Q|} \right)^p$, if $Q = Q_j^k$ for some (k, j) where $E(Q)$ denotes the corresponding set E_j^k associated to Q_j^k, and $a_Q = 0$ otherwise. If we apply the Carleson embedding to these a_Q, we will find the desired result provided that

$$\sum_{Q \subset R} a_Q \leq A \int_R \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx, \; R \in \mathcal{D}.$$

For $R \in \mathcal{D}$, we obtain

$$\begin{aligned}
\sum_{Q\subset R} a_Q &= \sum_{Q_j^k\subset R} v(E_j^k)\left(\prod_{i=1}^m \frac{\sigma_i(Q_j^k)}{|Q_j^k|}\right)^p \\
&= \sum_{Q_j^k\subset R} \int_{E_j^k}\left(\prod_{i=1}^m \frac{\sigma_i(Q_j^k)}{|Q_j^k|}\right)^p v(x)dx \\
&\le \sum_{Q_j^k\subset R} \int_{E_j^k} (\mathcal{M}(\overrightarrow{\sigma\chi_R}))^p v(x)dx \\
&\le [v,\vec{w}]_{S_{\vec{P}}}^p \prod_{i=1}^m \sigma_i(R)^{\frac{p}{p_i}} \\
&\le [v,\vec{w}]_{S_{\vec{P}}}^p [\vec{\omega}]_{RH_{\vec{P}}} \int_R \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx,
\end{aligned}$$

where in the next to last inequality we have used the $S_{\vec{P}}$ condition and in the last inequality we have used the $RH_{\vec{P}}$ condition. Thus, by Lemma 1.1 we get the desired result and the proof is complete.

□

Proof of Theorem 1.5. To prove this result we proceed using the standard argument as before. We obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathcal{M}^d(\overrightarrow{f\sigma})^p v dx &\le a^p \sum_{k,j}\left(\prod_{i=1}^m \frac{1}{|Q_j^k|}\int_{Q_j^k}|f_i|\sigma_i dy_i\right)^p v(Q_j^k) \\
&= a^p \sum_{k,j} v(Q_j^k)\left(\prod_{i=1}^m \frac{\sigma_i(Q_j^k)}{|Q_j^k|}\right)^p \left(\prod_{i=1}^m \frac{1}{\sigma_i(Q_j^k)}\int_{Q_j^k}|f_i|\sigma_i dy_i\right)^p \\
&\le a^p[v,\vec{\sigma}]_{B_{\vec{P}}} \sum_{k,j} |Q_j^k| \exp\left(\frac{1}{|Q_j^k|}\int_{Q_j^k}\log\prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx\right) \\
&\times \left(\prod_{i=1}^m \frac{1}{\sigma_i(Q_j^k)}\int_{Q_j^k}|f_i|\sigma_i dy_i\right)^p
\end{aligned}$$

And it follows

$$\int_{\mathbb{R}^n} \mathcal{M}^d(\overrightarrow{f\sigma})^p v dx \le a^p[v,\vec{\sigma}]_{B_{\vec{P}}} \sum_{Q\in\mathcal{D}} a_Q \left(\prod_{i=1}^m \frac{1}{\sigma_i(Q_j^k)}\int_{Q_j^k}|f_i|\sigma_i dy_i\right)^p,$$

where in next to last inequality we have used the $B_{\vec{P}}$ condition and

$$a_Q = |Q| \exp\left(\frac{1}{|Q|}\int_Q \log \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx\right),$$

if $Q = Q_j^k$ for some (k, j) and $a_Q = 0$, otherwise.

Next if we apply Carleson embedding to these a_Q, we obtain that (1.17) holds provided that

$$\sum_{Q\subset R} a_Q \leq A \int_R \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx,\ R \in \mathcal{D}.$$

For $R \in \mathcal{D}$, we have

$$\begin{aligned}
\sum_{Q\subset R} a_Q &\leq \sum_{Q_j^k\subset R} |Q_j^k| \exp\left(\frac{1}{|Q_j^k|}\int_{Q_j^k} \log \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx\right)\\
&\leq 2\sum_{Q_j^k} |E_j^k| \exp\left(\frac{1}{|Q_j^k|}\int_{Q_j^k} \log \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx\right)\\
&\leq 2\sum_{Q_j^k\subset R} \int_{E_j^k} M_0\left(\prod_{i=1}^m \sigma_i^{\frac{p}{p_i}}\chi_R\right) dx\\
&\leq 2\int_{\mathbb{R}^n} M_0\left(\prod_{i=1}^m \sigma_i^{\frac{p}{p_i}}\chi_R\right) dx\\
&\leq 2e\int_R \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}},
\end{aligned}$$

where M_0 is the (dyadic) logarithmic maximal function described in [6, Lemma 2.1] and also discussed in [15]. Here we have used that M_0 is bounded from L^1 into itself, and this concludes the proof of (1.17). □

Proof of Theorem 1.6. Proceeding as we did in the previous theorems, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathcal{M}^d(\overrightarrow{f\sigma})^p v dx &\leq a^p \sum_{k,j} v(Q_j^k)\left(\prod_{i=1}^m \frac{\sigma_i(Q_j^k)}{|Q_j^k|}\right)^p \left(\prod_{i=1}^m \frac{1}{\sigma_i(Q_j^k)}\int_{Q_j^k} |f_i|\sigma_i dy_i\right)^p\\
&\leq a^p[v,\vec{w}]_{A_{\vec{P}}} \sum_{Q\in\mathcal{D}} a_Q \left(\prod_{i=1}^m \frac{1}{\sigma_i(Q_j^k)}\int_{Q_j^k} |f_i|\sigma_i dy_i\right)^p,
\end{aligned}$$

where we have used the $A_{\vec{P}}$ condition and we have denoted by a_Q the following numbers

$$a_Q = \prod_{i=1}^{m} \sigma_i(Q)^{\frac{p}{p_i}},$$

if $Q = Q_j^k$ for some (j,k), and $a_Q = 0$, otherwise. Therefore it suffices to check that (1.11) holds for every $R \in \mathcal{D}$. Indeed,

$$\begin{aligned}
\sum_{Q \subset R} a_Q = \sum_{Q_j^k \subset R} \prod_{i=1}^{m} \sigma_i(Q_j^k)^{\frac{p}{p_i}} &= \sum_{Q_j^k \subset R} \prod_{i=1} \left(\frac{\sigma_i(Q_j^k)}{|Q_j^k|} \right)^{\frac{p}{p_i}} |Q_j^k| \\
&\le 2 \sum_{Q_j^k \subset R} |E_j^k| \prod_{i=1} \left(\frac{\sigma_i(Q_j^k)}{|Q_j^k|} \right)^{\frac{p}{p_i}} \\
&\le 2 \sum_{Q_j^k \subset R} \int_{E_j^k} \prod_{i=1}^{m} M(\sigma_i \chi_R)^{\frac{p}{p_i}} dx \\
&\le 2 \int_R \prod_{i=1}^{m} M(\sigma_i \chi_R)^{\frac{p}{p_i}} dx \\
&\le 2[\vec{\sigma}]_{W_{\vec{P}}^{\infty}} \int_R \prod_{i=1}^{m} \sigma_i^{\frac{p}{p_i}} dx,
\end{aligned}$$

where E_j^k are the sets associated with the cubes Q_j^k and we have used that $\vec{\sigma} \in W_{\vec{P}}^{\infty}$. Therefore we have proved (1.18).

$\square$

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